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Some applications of the Cauchy integral in problems of the deformation of thin plates☆

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ABSTRACT

Three complex potentials are used to obtain exact solutions of problems of the deformation of infinite thin plates of constant thickness with a circular or elliptical hole under uniaxial tension or under the action of internal pressure in three-dimensional setting. Solutions are sought using the Cauchy integral, which enables the solutions to be presented in finite form. The solutions obtained are compared with known solutions.

1. Plane problems of elasticity theory

Consider a thin plate of constant thickness, the mid-plane of which coincides with the OX_1X_2 plane of a rectangular Cartesian system of coordinates $OX_1X_2X_3$. Problems of the deformation of such plates by forces applied parallel and symmetrical to the mid-plane are treated in elasticity theory as plane or generalized plane stress state problems.[1,2]

In plane stress state problems, the main assumptions are that the stress tensor components $\sigma_{13}=\sigma_{23}=\sigma_{33}=0$, and that σ_{11}, σ_{22} and σ_{12} are independent of x_3. It follows from these assumptions, in particular, that the strain tensor components ε_{11}, ε_{22}, ε_{33} and ε_{12} are independent of x_3, while ε_{13} and ε_{23} are zero. The system of equations under these assumptions, including, in particular, one of the conditions of compatibility, does not guarantee that the remaining five conditions of compatibility are satisfied.[1]

All the conditions of compatibility are satisfied if the components of the displacement vector $\mathbf{u}=(u_1, u_2, u_3)$, irrespective of the relation between the stress and strain tensor components, are represented by formulae of the type[2]

$$u_1 = -A\frac{x_3^2}{2}+\omega_1(x_1,x_2), \quad u_2 = -B\frac{x_3^2}{2}+\omega_2(x_1,x_2), \quad u_3 = (Ax_1+Bx_2+C)x_3 \tag{1.1}$$

where A, B and C are certain constants.

The conclusion that the strain tensor component ε_{33} depends linerly on x_1, x_2, as follows from Eqs (1.1), is obviously more the exception than the rule in plane problems of elasticity.[1] Furthermore, if the condition $\sigma_{33}=0$ is adopted, we obtain the relation[2]

$$\lambda\left(\frac{\partial\omega_1}{\partial x_1}+\frac{\partial\omega_2}{\partial x_2}\right)+(\lambda+2\mu)(Ax_1+Bx_2+C) = 0$$

where λ and μ are the Lamé constants. This relation imposes constraints on the nature of the change in the functions $\omega_1(x_1, x_2)$ and $\omega_2(x_1, x_2)$ in the case of a plane stress state in a linearly elastic body.[2]

It was assumed in Ref 3 that, when plates of constant or variable thickness are deformed, the components of the displacement vector are specified in the form

$$u_1 = u_1(x_1,x_2), \quad u_2 = u_2(x_1,x_2), \quad u_3 = g(x_1,x_2)x_3 \tag{1.2}$$

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Here, we have for the strain tensor and stress tensor components

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{33} = g$$
$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \quad \varepsilon_{13} = \frac{1}{2}\frac{\partial g}{\partial x_1}x_3, \quad \varepsilon_{23} = \frac{1}{2}\frac{\partial g}{\partial x_2}x_3 \tag{1.3}$$

$$\sigma_{11} = \lambda\Theta + 2\mu\frac{\partial u_1}{\partial x_1}, \quad \sigma_{22} = \lambda\Theta + 2\mu\frac{\partial u_2}{\partial x_2}, \quad \sigma_{33} = \lambda\Theta + 2\mu g; \quad \Theta = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + g$$
$$\sigma_{12} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right), \quad \sigma_{13} = \mu\frac{\partial g}{\partial x_1}x_3, \quad \sigma_{23} = \mu\frac{\partial g}{\partial x_2}x_3 \tag{1.4}$$

When using functions of a complex variable, this approach led to the introduction of a complex potential that was supplementary to the two complex Kolosov–Muskhelishvili potentials.[4] Expressions for the quantities (1.2) to (1.4) were given in terms of three complex potentials, and it was established[3] that their use enables one to determine the stress tensor components that satisfy all three equilibrium equations and all Beltrami–Mitchell conditions of compatibility. This was also demonstrated for the specific example of the problem of the uniaxial stretching of an infinite thin plate with a free circular hole.[5]

Unfortunately, this approach also proved to have shortcomings. For example, in the problem examined, under the condition $\sigma_{33} = 0$, shear stresses σ_{13} and σ_{23} occurred.[5] Note, however, that their absolute values were considerably smaller than the absolute values of the other components, for example σ_{11} (for stretching along the OX_1 axis).

The appearance of the components σ_{13} and σ_{23} in such problems can be attributed, in particular, to distortion of the faces of the plate, which is especially marked in zones of high stress concentration. This conclusion is confirmed indirectly by data[6] on the stress distribution at the tip of a crack, represented as a narrow elliptical hole, where the distortion of the faces, as with the absolute values of the components σ_{13} and σ_{23}, was more significant.

It must also be pointed out that the components σ_{13} and σ_{23} depend linearly on x_3, by virtue of which their average values over the thickness of the plate are zero. This means that the presence of these stress components has no effect on the static equilibrium of the deformed plate.

2. The use of the Cauchy integral when employing three complex potentials in problems of the deformation of thin plates

Suppose L is a simple closed smooth contour in the complex plane $z = x_1 + ix_2$, S^+ is the finite part of the plane, bounded by the contour L, and S^- is the infinite part of the plane, consisting of points located outside L. The positive direction when passing around the contour will be the direction that keeps the region S^+ on the left.[7]

Cauchy integrals can be used most simply when solving plane problems of elasticity theory for regions that are mapped conformally onto a circle by rational functions. We will therefore assume that a certain region S of the z plane, bounded by a single simple closed contour L, is mapped onto the circle $|\zeta| < 1$ or onto the exterior of such a circle of the ζ plane by the relation

$$z = \omega(\zeta) \tag{2.1}$$

The circumference $|\zeta| = 1$ will be denoted by γ.

When two complex potentials $\varphi(z)$ and $\psi(z)$ are used,[4] the boundary conditions of the principal problems of elasticity theory are formulated using the relations

$$\varphi_1(\zeta) + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi_1'(\zeta)} + \overline{\psi_1(\zeta)} = p_1 + ip_2 \tag{2.2}$$

$$\kappa\varphi_1(\zeta) - \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi_1'(\zeta)} - \overline{\psi_1(\zeta)} = 2\mu(g_1 + ig_2) \tag{2.3}$$

(using the accepted terminology[4]). Here and below, we introduce the following notation

$$\varphi_1(\zeta) = \varphi[\omega(\zeta)], \quad \psi_1(\zeta) = \psi[\omega(\zeta)], \quad \frac{\varphi_1'(\zeta)}{\omega'(\zeta)} = \Phi_1(\zeta), \quad \frac{\psi_1'(\zeta)}{\omega'(\zeta)} = \Psi_1(\zeta)$$

The right-hand sides of relations (2.2) and (2.3) are determined by the principal force vector applied to the contour γ from the side of the positive normal in the first case, and by the specified boundary values of the displacement vector in the second. The coefficient κ is defined in terms of Poisson's ratio by the relation $\kappa' = 3–4\nu$; in the case of a thin plate (the generalized plane stress state) it is defined by the relation $\kappa = (3-\nu)/(1+\nu)$.[4]

The boundary condition of the first principal problem can be expressed slightly differently, using a relation that, using the conformal mapping (2.1), takes the form

$$\sigma_{\rho\rho} - i\sigma_{\rho\vartheta} = \Phi_1(\zeta) + \overline{\Phi_1(\zeta)} - \frac{\zeta^2\overline{\omega'(\zeta)}}{\rho^2\overline{\omega'(\rho)}}\left[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\Phi_1'(\zeta) + \Psi_1(\zeta)\right] \tag{2.4}$$

When the three complex potentials $\varphi(z)$, $\psi(z)$ and $f(z)$ are used, the combinations presented in relations (2.2) to (2.4) change their form. In particular, instead of (2.3), we have[3]

$$f_1(\zeta) - 2\left[\varphi_1(\zeta) + \frac{\omega(\zeta)}{\overline{\omega'(\zeta)}}\overline{\varphi_1'(\zeta)} + \overline{\psi_1(\zeta)}\right] = 4\mu(g_1 + ig_2) \tag{2.5}$$

Here, the cases of plane strain and a plane stress state differ in the specification of the corresponding conditions characterizing these states (for example, for the plane stress state, $\sigma_{33} = 0$).

The combination $\sigma_{\rho\rho} - i\sigma_{\rho\vartheta}$, taking relation (2.1) into account, is expressed in terms of the three complex potentials as follows:

$$\sigma_{\rho\rho} - i\sigma_{\rho\vartheta} = \frac{1}{4}[F_1(\zeta) + \overline{F_1(\zeta)}] - (1-2\nu)[\Phi_1(\zeta) + \overline{\Phi_1(\zeta)}] - \\ - \frac{\zeta^2\omega'(\zeta)}{\rho^2\overline{\omega'(\zeta)}}\left[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\Phi_1'(\zeta) + \Psi_1(\zeta)\right]; \quad F_1(\zeta) = \frac{f_1(\zeta)}{\omega'(\zeta)} \tag{2.6}$$

As regards relation (2.2), which is used to represent the boundary conditions, note that it was obtained[4] by forming a total differential. Unfortunately, when three complex potentials are used, this is not possible,[3] and therefore there is no correponding analogue of formula (2.2) here.

Obviously, by using relations (2.5) and (2.6), a certain group of plane problems of elasticity theory can be solved. In accordance with the selection made above, we will dwell on problems of the deformation of thin infinite plates of constant thickness (in the initial state) that have a single contour on which stresses of given form are specified. We will consider the particular features of the use of Cauchy integrals in specific examples.

3. The uniaxial stretching of an infinite plate with a free circular hole

Consider a plate of constant thickness that has a hole in the shape of a circle of radius R. We will assume that the stress P is specified at infinity; introducing a rectangular Cartesian system of coordinates $OX_1X_2X_3$ with its origin at the centre of the circular hole, and with the coordinate plane OX_1X_2 coinciding with the mid-plane of the plate, the stress P will be denoted by σ_{11}. The normal and shear stresses on the contour of the circular hole will be assumed to be zero.

We will assume that the principal normal stress on the faces of the plate is zero. For a plate having plane-parallel faces in the initial state, this assumption is equivalent to the condition

$$\sigma_{33} = 0 \tag{3.1}$$

Using of such an approximate boundary condition, an analytical solution was obtained in Ref. 5 with the aid of power series. The solution of the same problem using the Cauchy integral is undoubtedly of interest.

Assuming, in the given case, that

$$z = \omega(\zeta) = R\zeta \tag{3.2}$$

we reduce relation (2.6) to the following form

$$\sigma_{\rho\rho} - i\sigma_{\rho\vartheta} = (1+\nu)[\Phi_1(\zeta) + \overline{\Phi_1(\zeta)}] - \zeta^2\rho^{-2}[\bar{\zeta}\Phi_1'(\zeta) + \Psi_1(\zeta)] \tag{3.3}$$

The function $f(z)$ here was replaced by the function $\varphi(z)$ in accordance with the relation[8]

$$\sigma_{33} = 4(2-\nu)[\varphi'(z) + \overline{\varphi'(z)}] - [f'(z) + \overline{f'(z)}] = 0 \tag{3.4}$$

which follows from condition (3.1).

By virtue of the presence at infinity of non-zero stress tensor components, we specify the complex potentials in the form

$$\Phi_1(\zeta) = A_0 + \Phi_0(\zeta), \quad \Psi_1(\zeta) = B_0 + \Psi_0(\zeta) \tag{3.5}$$

where A_0 and B_0 are certain constants, and $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$ are functions that are holomorphic when $|\zeta| > 1$.

To determine the form of the complex potentials $\Phi_1(\zeta)$ and $\Psi_1(\zeta)$, we will use the boundary condition on the free contour $\gamma:|\zeta| = 1$, in accordance with which, from Eq. (3.3), we obtain

$$(1+\nu)[\Phi_1(t) + \overline{\Phi_1(t)}] - t\Phi_1'(t) - t^2\Psi_1(t) = 0 \tag{3.6}$$

where $t = e^{i\vartheta}$ is an arbitrary point of the contour γ.

We will divide Eq. (3.6) by t^2 and change to conjugate quantities

$$(1+\nu)[t^2\Phi_1(t) + t^2\overline{\Phi_1(t)}] - t\overline{\Phi_1'(t)} - \overline{\Psi_1(t)} = 0 \tag{3.7}$$

It can be seen that the functions $t^2\overline{\Phi_1(t)}$, $t\overline{\Phi_1'(t)}$ and $\overline{\Psi_1(t)}$ are respectively the boundary values of the functions $\zeta^2\overline{\Phi_1}(\zeta^{-1})$, $\zeta\overline{\Phi_1'}(\zeta^{-1})$ and $\overline{\Psi_1}(\zeta^{-1})$, which are holomorphic within the contour γ. By virtue of this, using the properties of the Cauchy integral, with $\zeta \in S^-$, from the relation

$$(1+\nu)\left[\frac{1}{2\pi i}\int_{\gamma}\frac{t^2\Phi_1(t)}{t-\zeta}dt+\frac{1}{2\pi i}\int_{\gamma}\frac{t^2\overline{\Phi_1(t)}}{t-\zeta}dt\right]-\frac{1}{2\pi i}\int_{\gamma}\frac{t\overline{\Phi_1'(t)}}{t-\zeta}dt-\frac{1}{2\pi i}\int_{\gamma}\frac{\overline{\Psi_1(t)}}{t-\zeta}dt = 0$$

we obtain

$$(1+\nu)[-\zeta^2\Phi_1(\zeta)+A_0\zeta^2+A_1\zeta+A_2] = 0$$

Hence, we have

$$\Phi_1(\zeta) = A_0+A_1\zeta^{-1}+A_2\zeta^{-2} \tag{3.8}$$

To determine $\Psi_1(\zeta)$, we will use relation (3.6). The function $\Psi_1(t)$ is the boundary value of a function that is holomorphic outside the contour γ and equal to B_0 when $\zeta=\infty$. The nature of the remaining functions in Eq. (3.6) multiplied by t^{-2} is established just as easily. Using the properties of the Cauchy integral, we can integrate this last equation over the contour γ with $\zeta\in S^-$; we obtain

$$\Psi_1(\zeta) = (1+\nu)[\zeta^{-2}\Phi_1(\zeta)+\overline{A}_0\zeta^{-2}+\overline{A}_1\zeta^{-1}+\overline{A}_0]-\zeta^{-1}\Phi_1'(\zeta)+B_0 \tag{3.9}$$

To determine the constants A_0 and B_0, we use the boundary condition at infinity. This reduces to expressions of the form

$$\sigma_{11}+\sigma_{22}|_\infty = P,\quad \sigma_{22}-\sigma_{11}+2i\sigma_{12}|_\infty = -P$$

Representing these combinations in terms of complex potentials[3,4,8]

$$\sigma_{11}+\sigma_{22} = 2(1+\nu)[\Phi_1(\zeta)+\overline{\Phi_1(\zeta)}]$$

$$\sigma_{22}-\sigma_{11}+2i\sigma_{12} = 2[\overline{\omega(\zeta)}\Phi_1'(\zeta)+\Psi_1(\zeta)]$$

we have

$$4(1+\nu)\mathrm{Re}\Phi_1(\zeta)|_\infty = P,\quad 2[\bar{\zeta}\Phi_1'(\zeta)+\Psi_1(\zeta)]_\infty = -P$$

Hence, we obtain

$$A_{01} = \frac{P}{4(1+\nu)},\quad B_0 = -\frac{P}{2} \tag{3.10}$$

where A_{01} is the real part of the coefficient A_0. Since there is no rotation at infinity, the imaginary part of the coefficient A_0 is zero,[4] and consequently $A_0=A_{01}$.

The coefficient A_1 can be determined from the condition for the displacements to be unique[3]

$$A_1-2\overline{B}_1 = 0 \tag{3.11}$$

Here B_1 is the coefficient of the expansion for ζ^{-1} of the function $\Psi_1(\zeta)$. In the general case, a second equation is needed to determine it. We obtain this equation using Eq. (3.6); multiplying it by $(2\pi it^2)^{-1}$, and integrating over the contour γ, we have $(1+\nu)\overline{A_1}-B_1=0$. From this, and from Eq. (3.11), it follows that

$$A_1 = 0,\quad B_1 = 0$$

We will determine the constant A_2 by integrating (over the contour γ) Eq. (3.6) multiplied by $(2\pi it^3)^{-1}$, as a result of which we obtain $(1+\nu)\overline{A_2}-B_0=0$, or

$$A_2 = -\frac{P}{2(1+\nu)}$$

Hence, from relations (3.8) and (3.9) we find

$$\Phi_1(\zeta) = \frac{P}{4(1+\nu)}-\frac{P}{2(1+\nu)}\frac{1}{\zeta^2},\quad \Psi_1(\zeta) = -\frac{P}{2}+\frac{P}{2}\frac{1}{\zeta^2}-\frac{3+\nu}{1+\nu}\frac{P}{2}\frac{1}{\zeta^4}$$

In the given case it is easy to change to the variable z, assuming that $\varsigma = z/R$. Using the expressions obtained for the complex potentials, we find the solution of the problem under consideration in the cylindrical system of coordinates (r, α, x_3):

$$\sigma_{rr} = \frac{P}{2}\left[\left(1-\frac{R^2}{r^2}\right)+\left(1-2\frac{2+\nu}{1+\nu}\frac{R^2}{r^2}+\frac{3+\nu}{1+\nu}\frac{R^4}{r^4}\right)\cos 2\alpha\right]$$

$$\sigma_{\alpha\alpha} = \frac{P}{2}\left[\left(1+\frac{R^2}{r^2}\right)-\left(1+2\frac{\nu}{1+\nu}\frac{R^2}{r^2}+\frac{3+\nu}{1+\nu}\frac{R^4}{r^4}\right)\cos 2\alpha\right]$$

$$\sigma_{r\alpha} = -\frac{P}{2}\left(1+\frac{2}{1+\nu}\frac{R^2}{r^2}-\frac{3+\nu}{1+\nu}\frac{R^4}{r^4}\right)\sin 2\alpha$$

The angle α is measured from the positive direction of the OX_1 axis. Using the relations[3]

$$\sigma_{\rho 3} - i\sigma_{\vartheta 3} = -2\nu x_3\frac{\zeta}{\rho}\frac{1}{|\omega'(\zeta)|}\Phi_1'(\zeta)$$

we find the shear stress components associated with the third coordinate:

$$\sigma_{r3} = -2P\frac{\nu}{1+\nu}\frac{R^2 x_3}{r^3}\cos 2\alpha, \quad \sigma_{\alpha 3} = -2P\frac{\nu}{1+\nu}\frac{R^2 x_3}{r^3}\sin 2\alpha$$

It is easily verified that the solution of the problem satisfies all the equilibrium equations and conditions of compatibility, as well as the above boundary conditions. Note that, when $\nu = 0$, the solution is identical with the solution of Kirsch's problem.[1]

4. The uniaxial stretching of a plate with a free elliptical hole

We will consider an infinite thin plate of constant thickness with a free elliptical hole. At infinity the plate is loaded by an evenly distributed tensile force, the direction of which makes an angle β with the semi-major axis of the ellipse.

The region outside the elliptical hole in the complex z plane can be mapped onto the exterior of the unit circle in the complex ζ plane by means of the function

$$z = \omega(\zeta) = R\left(\zeta + \frac{m}{\zeta}\right), \quad R > 0, \quad 0 \le m < 1 \tag{4.1}$$

In this case, the circumference $\gamma: |\zeta| = 1$ has a corresponding ellipse with its centre at the origin of coordinates and with the semi-axes

$$a = R(1+m), \quad b = R(1-m)$$

Hence

$$R = \frac{a+b}{2}, \quad m = \frac{a-b}{a+b}$$

We will assume that the semi-major axis of the ellipse coincides with the OX_1 axis. Then, an evenly distributed tensile stress P acts at an angle β to the OX_1 axis. By virtue of this

$$\sigma_{11} + \sigma_{22} = P, \quad \sigma_{22} - \sigma_{11} + 2i\sigma_{12} = -Pe^{-2i\beta} \tag{4.2}$$

Since, in this problem, there are no stresses on the contour of the elliptical hole, the boundary condition on the contour γ can be represented in the form[4,5]

$$\sigma_{\rho\rho} - i\sigma_{\rho\vartheta}\big|_\gamma = \left\{(1+\nu)[\Phi_1(\zeta)+\overline{\Phi_1(\zeta)}] - \frac{\zeta^2\omega'(\zeta)}{|\zeta|^2\overline{\omega'(\zeta)}}\left[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\Phi_1'(\zeta)+\Psi_1(\zeta)\right]\right\}\Bigg|_\gamma = 0$$

Taking condition (4.1) into account, we have on the contour of the circular hole $|\zeta| = 1$

$$(1+\nu)(1-mt^2)[\Phi_1(t)+\overline{\Phi_1(t)}] - (t+mt^3)\Phi_1'(t) - (t^2-m)\Psi_1(t) = 0 \tag{4.3}$$

where $t = e^{i\vartheta}$ is an arbitrary point of the contour γ; here, we have taken into account that $t\bar{t} = 1$.

Bearing in mind the presence at infinity of non-zero stress tensor components, the complex potentials are given in the form

$$\Phi_1(\zeta) = A_0 + \Phi_0(\zeta), \quad \Psi_1(\zeta) = B_0 + \Psi_0(\zeta) \tag{4.4}$$

where A_0 and B_0 are certain constants, and $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$ are functions that are holomorphic when $|\zeta| > 1$.

When Cauchy integrals are used to solve the problem, it is more convenient to divide relation (4.3) is conveniently divided by t^2, after which we change to conjugate quantities. As a result we have

$$(1+\nu)(t^2-m)[\Phi_1(t)+\overline{\Phi_1(t)}]-\left(t+\frac{m}{t}\right)\overline{\Phi_1'(t)}-(1-mt^2)\overline{\Psi_1(t)} = 0 \tag{4.5}$$

Multiplying Eq. (4.5) by $(2\pi i(t-\zeta))^{-1}$, we integrate the expression obtained over the contour γ. Using the properties of the Cauchy integral, we obtain

$$\Phi_1(\zeta) = A_0+\frac{A_1\zeta+A_2}{\zeta^2-m}, \quad \zeta\in S^- \tag{4.6}$$

where A_1 and A_2 are the coefficients of the first two terms of the expansion of the function $\Phi_0(\zeta)$. By virtue of the fact that

$$\frac{1}{\zeta^2-m} = \frac{1}{\zeta^2}\left(1-\frac{m}{\zeta^2}\right)^{-1} = \frac{1}{\zeta^2}\left(1+\frac{m}{\zeta^2}+\frac{m^2}{\zeta^4}+\ldots\right) \tag{4.7}$$

from Eq. (4.6) we obtain

$$\Phi_1(\zeta) = A_0+\frac{A_1}{\zeta}+\frac{A_2}{\zeta^2}+\frac{mA_1}{\zeta^3}+\frac{mA_2}{\zeta^4}+\ldots \tag{4.8}$$

from which the form of these coefficients is established.

To determine the function $\Psi_1(\zeta)$, we will again use relation (4.3). Multiplying it by $(2\pi it^2(t-\zeta))^{-1}$, integrating the equation obtained over the contour γ, and using the properties of the Cauchy integral, we obtain

$$\Psi_1(\zeta) = \frac{1}{\zeta^2-m}\{(1+\nu)[(1-m\zeta^2)\Phi_1(\zeta)+mA_0\zeta^2+\overline{A_0}+\overline{A_1}\zeta]\}-$$
$$-\frac{1}{\zeta^2-m}\{(1+m\zeta^2)\zeta\Phi_1'(\zeta)+B_0\zeta^2\}, \quad \zeta\in S^- \tag{4.9}$$

We will now determine the coefficients in Eqs (4.6) and (4.9). As in Section 3, from the relations

$$\sigma_{11}+\sigma_{22} = 2(1+\nu)[\Phi(z)+\overline{\Phi(z)}] = 2(1+\nu)[\Phi_1(\zeta)+\overline{\Phi_1(\zeta)}]$$

$$\sigma_{22}-\sigma_{11}+2i\sigma_{12} = 2[\bar{z}\Phi'(z)+\Psi(z)] = 2\left[\frac{\overline{\omega(\zeta)}}{\omega'(\zeta)}\Phi_1'(\zeta)+\Psi_1(\zeta)\right]$$

and taking expressions (4.4) into account, we obtain

$$A_0 = \frac{P}{4(1+\nu)}, \quad B_0 = -\frac{P}{2}e^{-2i\beta} \tag{4.10}$$

The coefficients A_1 and A_2 are determined from two relations. One of them, connected with the condition for the displacements to be unique

$$\overline{A_1}-2B_1 = 0$$

has already been mentioned above, in Section 3. The second equation is found by multiplying Eq. (4.3) by $(2\pi it^2)^{-1}$ and then integrating the expression obtained over the contour γ:

$$\frac{1+\nu}{2\pi i}\int_\gamma\left(\frac{1}{t^2}-m\right)[\Phi_1(t)+\overline{\Phi_1(t)}]dt-\frac{1}{2\pi i}\int_\gamma\left(\frac{1}{t}+mt\right)\overline{\Phi_1'(t)}dt-\frac{1}{2\pi i}\int_\gamma\left(1-\frac{m}{t^2}\right)\overline{\Psi_1(t)}dt = 0$$

Bearing relations (4.7) and (4.8) in mind, and taking into account the possibility of representing the holomorphic function $\Psi_0(\zeta)$ (outside the contour γ) in the form

$$\Psi_0(\zeta) = \frac{B_1}{\zeta}+O\left(\frac{1}{\zeta^2}\right)$$

from the last equation, calculating the residues, we obtain

$$-(1+\nu)mA_1+(1+\nu)\overline{A_1}+mA_1-B_1 = 0$$

From these relations we find

$$A_1 = 0, \quad B_1 = 0$$

Finally, to determine the coefficient A_2, we multiply relation (4.3) by $(2\pi it^3)^{-1}$, after which we integrate the equation obtained over the contour γ. As a result, we have

$$(1+\nu)[m(A_0+\overline{A_0})-\overline{A_2}]+B_0 = 0$$

From this, taking Eqs (4.10) into account, we obtain

$$A_2 = \frac{B_0}{1+\nu}+2mA_0 = \frac{P}{2}\frac{m-e^{2i\beta}}{1+\nu}$$

Thus, Eqs (4.6) and (4.9) take the form

$$\Phi_1(z) = \frac{P}{4(1+\nu)}\frac{\zeta^2+m-2e^{2i\beta}}{\zeta^2-m} = \frac{P}{4(1+\nu)}\left[1+2\frac{m-e^{2i\beta}}{\zeta^2-m}\right]$$

$$\Psi_1(\zeta) = \frac{P}{2(\zeta^2-m)}\left\{1-\zeta^2e^{-2i\beta}+\frac{m-e^{2i\beta}}{\zeta^2-m}\left[1-m\zeta^2+\frac{2(1+m\zeta^2)\zeta^2}{(1+\nu)(\zeta^2-m)}\right]\right\}$$

The expressions for the complex potentials are extremely cumbersome, as are the individual expressions for the stress tensor components. For this reason, following a well-known approach,[4] we will represent the principal combinations of the stress components using the assumptions adopted above. For the sum of the normal stress components, we have

$$\sigma_{\rho\rho}+\sigma_{\vartheta\vartheta} = 2(1+\nu)[\Phi_1(\zeta)+\overline{\Phi_1(\zeta)}] = P\frac{\rho^4-2\rho^2\cos2(\vartheta-\beta)+2m\cos2\beta-m^2}{\Delta(\rho,\vartheta)}$$

$$\Delta(\rho,\vartheta) = \rho^4-2m\rho^2\cos2\vartheta+m^2 \tag{4.11}$$

The following combination is incomparably more cumbersome, and therefore we will limit ourselves here to the compact representation

$$\sigma_{\vartheta\vartheta}-\sigma_{\rho\rho}+2i\sigma_{\rho\vartheta} = \frac{2\zeta^2}{\rho^2\overline{\omega'(\zeta)}}[\overline{\omega(\zeta)}\Phi_1'(\zeta)+\omega'(\zeta)\Psi_1(\zeta)] =$$

$$= \frac{2P\rho^2}{\bar\zeta^2-m}\left\{\frac{m-e^{2i\beta}}{\zeta^2-m}\left[\frac{1+m\zeta^2-(\bar\zeta^2+m)e^{2i\vartheta}}{(1+\nu)(\zeta^2-m)}+\frac{1-m\zeta^2}{2\zeta^2}\right]+\frac{1}{2\zeta^2}-\frac{e^{-2i\beta}}{2}\right\} \tag{4.12}$$

From formulae (4.11) and (4.12) it can be seen that, when $m=0$, $\beta=0$ and $\vartheta=\pi/2$, on the contour of the hole the stress component $\sigma_{\vartheta\vartheta}=3P$, which corresponds to well-known results.[1,4]

The real part of relation (4.12), together with Eq. (4.11), forms a system of two equations in the stress components $\sigma_{\rho\rho}$ and $\sigma_{\vartheta\vartheta}$, while the imaginary part of relation (4.12) enables the shear stress component $\sigma_{\rho\vartheta}$ to be determined directly.

To determine the shear stress components connected with the third coordinate, we use a well-known relation[3] that, in this case, takes the form

$$\sigma_{\rho3}-i\sigma_{\vartheta3} = -2\nu x_3\frac{\zeta}{\rho}\frac{1}{|\omega'(\zeta)|}\Phi_1'(\zeta) =$$

$$= \frac{2\nu x_3P}{1+\nu}\frac{(m-e^{2i\beta})\zeta^2}{\rho|\omega'(\zeta)|(\zeta^2-m)^2} = \frac{2\nu x_3P}{(1+\nu)R}\frac{\rho(m-e^{2i\beta})\zeta^2(\bar\zeta^2-m)^2}{[\Delta(\rho,\vartheta)]^{5/2}} \tag{4.13}$$

Separate values of the stress components are established by separating the real and imaginary parts from Eq. (4.13).

We will give the relation, inverse to expression (4.1), between the variables used here

$$\zeta = \frac{z\pm\sqrt{z^2-4mR^2}}{2R}$$

in which the plus sign must be chosen, since to each point z outside the elliptical hole there corresponds a point of the region $|\zeta|>1$.

It has been established directly that, when $\nu=0$, the potentials obtained take a well-known form,[4] and when $m=0$ they are identical with the corresponding potentials given earlier.[5] It should be also noted that the numerical data obtained using the solution presented here correspond to data[6] obtained using the solution in the form of series.

5. The action of a uniform pressure on the contour of an elliptical hole in an infinite plate

We will consider an infinite thin plate of constant thickness with an elliptical hole. At infinity, there are no stresses, while a constant uniform pressure P is applied to the contour of the elliptical hole, leading to the appearance on it of a normal stress $-P$. On this basis, by means of conformal mapping (4.1), we will formulate the boundary-value problem on the contour of the circular hole $|\zeta| = 1$, writing it, after multiplication by $(1 - mt^2)$, in the form

$$(1+\nu)(1-mt^2)[\Phi_1(t)+\overline{\Phi_1(t)}]-(t+mt^3)\Phi_1'(t)-(t^2-m)\Psi_1(t) = -P(1-mt^2) \tag{5.1}$$

where $t = e^{i\vartheta}$ is an arbitrary point of the contour γ.

As there are no stresses at infinity, in relations (4.4) the constants $A_0 = B_0 = 0$, by virtue of which the complex potentials $\Phi_1(\zeta)$ and $\Psi_1(\zeta)$ are holomorphic functions.

We multiply relation (5.1) by t^{-2} and change to conjugate quantities, after which we will multiply the relation obtained by $(2\pi i(t-\zeta))^{-1}$. Integrating over the contour γ, we arrive at the integral relation

$$\frac{1+\nu}{2\pi i}\left[\int_\gamma \frac{t^2-m}{t-\zeta}\Phi_1(t)dt+\int_\gamma \frac{t^2-m}{t-\zeta}\overline{\Phi_1(t)}dt\right]-$$

$$-\frac{1}{2\pi i}\int_\gamma\left(t+\frac{m}{t}\right)\frac{\overline{\Phi_1'(t)}}{t-\zeta}dt-\frac{1}{2\pi i}\int_\gamma\frac{1-mt^2}{t-\zeta}\overline{\Psi_1(t)}dt = -\frac{P}{2\pi i}\int_\gamma\frac{t^2-m}{t-\zeta}dt$$

Using the properties of the Cauchy integral, we obtain

$$\Phi_1(\zeta) = \frac{A_1\zeta+A_2}{\zeta^2-m} \tag{5.2}$$

To determine the form of the second complex potential, we multiply relation (5.1) by $(2\pi i t^2(t-\zeta))^{-1}$, after which we will integrate it over the contour γ.

Using the properties of the Cauchy integral, we obtain

$$\Psi_1(\zeta) = \frac{P}{\zeta^2-m}+\frac{(1+\nu)[(1-m\zeta^2)\Phi_1(\zeta)-\overline{A_1}\zeta]}{\zeta^2-m}-\frac{(\zeta+m\zeta^3)\Phi_1'(\zeta)}{\zeta^2-m} \tag{5.3}$$

We will now determine the constants A_1 and A_2 in the expression for the complex potential $\Phi_1(\zeta)$. To do this, in relation (5.1) we change to conjugate quantities, as a result of which we have

$$(1+\nu)\left(1-\frac{m}{t^2}\right)[\Phi_1(t)+\overline{\Phi_1(t)}]-\left(\frac{1}{t}+\frac{m}{t^3}\right)\overline{\Phi_1'(t)}-\left(\frac{1}{t^2}-m\right)\overline{\Psi_1(t)} = -P\left(1-\frac{m}{t^2}\right) \tag{5.4}$$

Since there are no stresses at infinity and the consequent holomorphicity of the functions $\Phi_1(\zeta)$ and $\Psi_1(\zeta)$ outside the contour γ, including an infinitely remote point,[4] and also taking into account expansion (4.8), we have

$$\Phi_1(\zeta) = \frac{A_1}{\zeta}+\frac{A_2}{\zeta^2}+O\left(\frac{1}{\zeta^3}\right),\quad \Psi_1(\zeta) = \frac{B_1}{\zeta}+O\left(\frac{1}{\zeta^2}\right)$$

Integrating Eq. (5.4) over the contour γ, we obtain

$$(1+\nu)(A_1-m\overline{A_1})+m\overline{A_1}-\overline{B_1} = 0$$

Supplementing this equation with the condition for the displacements to be unique[3] $A_1 - 2\bar{B}_1 = 0$, we have

$$A_1 = 0,\quad B_1 = 0$$

To determine A_2, we multiply the left- and right-hand sides of Eq. (5.4) by t and again integrate over γ. As a result we obtain

$$A_2 = \frac{Pm}{1+\nu}$$

Thus, we obtain the following expressions for the complex potentials

$$\Phi_1(\zeta) = \frac{Pm}{(1+\nu)(\zeta^2-m)}$$

$$\Psi_1(\zeta) = \frac{P\zeta^2}{(\zeta^2-m)^3}\left[(\zeta^2-m)(1-m^2)+\frac{2m(1+m\zeta^2)}{1+\nu}\right]$$

On the assumption that $\nu = 0$, the potentials presented here are identical with those given earlier.[4]

As in Section 4, we will give expressions for the main combinations of the stress tensor components defined by relations (4.11) and (4.12):

$$\sigma_{\rho\rho}+\sigma_{\vartheta\vartheta} = 2mP\frac{\zeta^2+\bar{\zeta}^2-2m}{(\zeta^2-m)(\bar{\zeta}^2-m)}$$

$$\sigma_{\vartheta\vartheta}-\sigma_{\rho\rho}+2i\sigma_{\rho\vartheta} = \frac{2P\rho^2}{(\zeta^2-m)(\bar{\zeta}^2-m)^2}\times$$

$$\times\left\{\frac{2m\lfloor(1+m\zeta^2)-(\bar{\zeta}^2+m)e^{2i\vartheta}\rfloor}{1+\nu}+(\zeta^2-m)(1-m)\right\}$$

$$\sigma_{\rho 3}-i\sigma_{\vartheta 3} = \frac{4\nu mPx_3}{1+\nu}\frac{\zeta^2}{|\omega'(\zeta)|(\zeta^2-m)^2} = \frac{4\nu mPx_3}{(1+\nu)R}\frac{\rho\zeta^2}{(\zeta^2-m)^2\sqrt{\Delta(\rho,\vartheta)}}$$

In this problem the stress distributions along the line of symmetry when $\vartheta=0$ are pf greatest interest. We will give these, especially as the expressions for σ_{11} and σ_{22} were misprinted in Ref. 4 (p. 308). In the case considered here, when $\vartheta=0$, we have

$$\begin{Bmatrix}\sigma_{\rho\rho}\\ \sigma_{\vartheta\vartheta}\end{Bmatrix} = \frac{P}{\rho^2-m}\left[2m\mp\frac{\rho^2(1-m^2)}{\rho^2-m}\pm\frac{m(1-m)}{1+\nu}\frac{\rho^2(\rho^2-1)}{(\rho^2-m)^2}\right]$$

It can be seen that for any value of $m\in[0,1)$

$$\sigma_{\rho\rho}|_{\rho=1} = -P,\quad \sigma_{\vartheta\vartheta}|_{\rho=1} = P\frac{1+3m}{1-m}$$

And, finally, when $m=0$, we have the well-known result

$$\sigma_{\rho\rho}|_{\rho=1} = -\sigma_{\vartheta\vartheta}|_{\rho=1} = -P$$

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